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ARE ALL SIMPLE 4-POLYTOPES HAMILTONIAN?

BY

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ABSTRACT

We construct an extensive family of non-Hamiltonian, 4-regular, 4-connected graphs and show that none of these graphs is the graph of a simple 4-polytope.

1. Introduction

A graph G is d-polytopal if it is isomorphic to the 1-skeleton of a d-polytope. A d-polytope is simple, if every vertex belongs to exactly d facets (d edges). The graph of a simple d-polytope is d-regular and d-connected. Many problems concerning d-polytopal graphs were formulated over 20 years ago. In spite of that, very little progress was made on most of these problems. In this paper, we try to shed more light on one of these problems: Barnette's conjecture. D. Barnette has conjectured that every simple 4-polytope is Hamiltonian (see Grünbaum [3], p. 1145). Since no characterization of 4-polytopal graphs is known, it seems that there are two possible alternatives to try to resolve Barnette's conjecture:

(i) Construct families of simple 4-polytopes and check their Hamiltonicity.

(ii) Construct 4-regular, 4-connected, non-Hamiltonian graphs, and check their polytopality.

With luck, either (i) will produce a non-Hamiltonian simple 4-polytope, or (ii) will produce a polytopal graph. (At this point, the author wishes to express his doubts in the validity of Barnette's conjecture.) Due to the difficulties of checking the Hamiltonicity or polytopality of graphs, these seem like the devil's alternative, but they seem like the only alternatives available at this point.

The first alternative was adopted in Rosenfeld and Barnette [7]. In this paper (with the later arrival of the Four Color Theorem), it was shown that all simple

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4-prisms are Hamiltonian. V. Klee [4] proved that the duals of the cyclic 4-polytopes $C(n;4)$ are Hamiltonian. Grünbaum, Kleinschmidt and Rosenfeld (in preparation) extended Klee's result to a large family of neighborly 4-polytopes. In this paper we present an attempt to use the second alternative. We construct an extensive family of 4-regular, 4-connected non-Hamiltonian graphs (these are all such graphs known to the author) and show that none of these graphs is polytopal. To prove this we use the existence of an infinite family of forbidden subgraphs in simple 4-polytopal graphs that we derive. As a byproduct of our attempt, we obtain a 4-regular, 4-connected, non-Hamiltonian graph with 54 vertices. The smallest (and first) counter example to Nash-Williams' conjecture (that all 4-regular 4-connected graphs are Hamiltonian), the Meredith graph, has 70 vertices (Meredith [6]).

2. Definitions and preliminaries

We use the standard terminology and notation of graph theory (Bondy and Murty [1]) and convex polytopes (Grünbaum [2]). If P is a simple 4-polytope, then its boundary complex, $\mathscr{B}(P)$, has the following properties:

(a) The facets of P (the 3-faces) are simple 3-polytopes.

(b) If k facets have a non-empty intersection, then the dimension of their intesection is $4-k$.

(c) Every $(4-k)$-face of P is the intersection of exactly k facets.

(d) If a k-face H meets a facet F, $H \not\subseteq F$, then $F \cap H$ is a $(k-1)$-face.

These are standard results; for details, see Grünbaum [2], ch. 3. It follows that if a 4-regular, 4-connected graph is 4-polytopal, it must have the following properties:

(1) G contains a family of subgraphs $\{G_1, \cdots, G_n\}$, each a cubic 3-connected, planar graph. (We call these subgraphs the "facets" of G.)

(2) Each vertex of G is contained in exactly 4 "facets" of G, and each edge of G is contained in exactly three "facets".

(3) $G_i \cap G_j$ is either empty, or a two-face of G_i and G_j, that is, an induced cycle in the graph G.

(4) Any two-face of G_i is the intersection of exactly two "facets".

It should be noted that if G satisfies (1)–(4), it does not necessarily mean that G is 4-polytopal, but the converse is always true.

We denote by $N(g)$ the set of vertices adjacent to g in a graph G. Let G and H be two disjoint graphs. Let $g \in V(G)$ and $h \in V(H)$ be two vertices in these graphs. We further assume that both G and H are 4-regular, 4-connected graphs. Let

$$N(g) = \{g_1, g_2, g_3, g_4\} \quad \text{and} \quad N(h) = \{h_1, h_2, h_3, h_4\}.$$

Let $G_1 = G \setminus \{g\} \cup H \setminus \{h\} \cup \{(g_1, h_1), (g_2, h_2), (g_3, h_3), (g_4, h_4)\}$. Obviously, G_1 is a 4-regular graph. It is easy to see that G_1 is also 4-connected. We say that G_1 is obtained from G by substituting the graph $H \setminus \{h\}$ for g.

3. 4-regular, 4-connected, non-Hamiltonian graphs

In [6], Meredith constructed r-regular, r-connected non-Hamiltonian graphs. His construction was based on substituting complete bipartite graphs for the vertices of a multigraph obtained by replacing some edges of the Peterson graph P by multiple edges. The Meredith graph thus obtained is a 4-regular, 4-connected non-Hamiltonian graph with 70 vertices. We construct below a family of 4-regular, 4-connected non-Hamiltonian graphs that contain this graph.

Let $\mathcal{H}_4$ denote the set of all 4-regular, 4-connected graphs G, such that for each G there is at least one vertex $g \in V(G)$, with $N(g) = \{g_1, g_2, g_3, g_4\}$, so that $V(G \setminus \{g\})$ cannot be covered by two disjoint paths with endpoints $\{g_1, g_2, g_3, g_4\}$. (For example, any 4-regular, 4-connected bipartite graph is in $\mathcal{H}_4$.) In the sequel, whenever we use a member G of $\mathcal{H}_4$, and a vertex $g \in V(G)$, we assume that g has the above mentioned property.

LEMMA 1. *Suppose that a subgraph K of a 4-regular, 4-connected graph H is isomorphic to $G \setminus \{g\}$ where $G \in \mathcal{H}_4$. Let H_1 be obtained from H by contracting K to a single vertex. H is Hamiltonian if and only if H_1 is.*

LEMMA 2. *Let H and H_1 be as in Lemma 1; H is 4-connected iff H_1 is 4-edge connected.*

The proofs of these lemmas are essentially similar to the proofs of theorems 1 and 3 in Meredith [6]; we omit the details.

LEMMA 3. *Let G be a 3-connected cubic graph. Let G^* be the multi-graph obtained from G by replacing each edge $e \in F$, where F is a given 1-factor of G, by a pair of parallel edges. Then G^* is 4-edge connected.*

PROOF. Since G is cubic and 3-connected, it is 3-edge connected. Assume that the lemma is false, and let $\{e_1, e_2, e_3\} \subseteq E(G^*)$ be an edge cut set of G^*. If $e_1 \in F$ and e_1^* (the edge parallel to e_1) belongs to $\{e_1, e_2, e_3\}$, say $e_1^* = e_2$, then $\{e_1, e_3\}$ would be an edge cut set in G. If $e_1^* \not\in \{e_2, e_3\}$, then $\{e_2, e_3\}$ would be an edge cut set in G. Since both contradict our assumption, we conclude that $\{e_1, e_2, e_3\} \cap F = \emptyset$. That means that the edges e_1, e_2, e_3 are in $E(G) \setminus F$, and they separate G. Since G is cubic and 3-connected, the only way that G can be separated by these edges is if the edges are pairwise disjoint, and $G \setminus \{e_1, e_2, e_3\}$ has exactly two components, G_1 and G_2 (Fig. 1).

Since F is a 1-factor of G, $F \cap G_1$ must be a 1-factor of G_1. But G_1 has exactly 3 vertices of degree two and since all the other vertices have degree three, G_1 has an odd number of vertices and therefore cannot have a 1-factor. This proves that G^* is 4-edge connected.

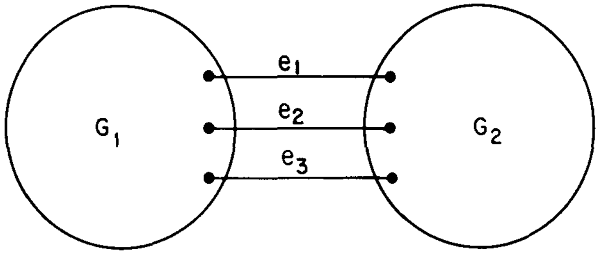

Fig. 1.

THEOREM 1. *Let G be a cubic, 3-connected, non-Hamiltonian graph. Let $F \subseteq E(G)$ be a 1-factor of G. Let G^* be the 4-regular multi-graph obtained by replacing each edge $e \in F$ by a pair of parallel edges. Let $\bar{G}$ be obtained from G^* by substituting for each vertex $g \in V(G^*)$ some graph $H(g)\setminus\{h\}$ where $H(g) \in \mathcal{H}_4$. Then $\bar{G}$ is a 4-regular 4-connected non-Hamiltonian graph.*

PROOF. $\bar{G}$ is obviously 4-regular. By Lemma 3, G^* is 4-edge connected. By Lemma 2, $\bar{G}$ is 4-connected. $\bar{G}$ is non-Hamiltonian by Lemma 1.

We denote by $\mathcal{M}_0$ the family of all 4-regular, 4-connected graphs $\bar{G}$ obtained by the above construction. Observe that the Meredith graph (Fig. 2) is obtained

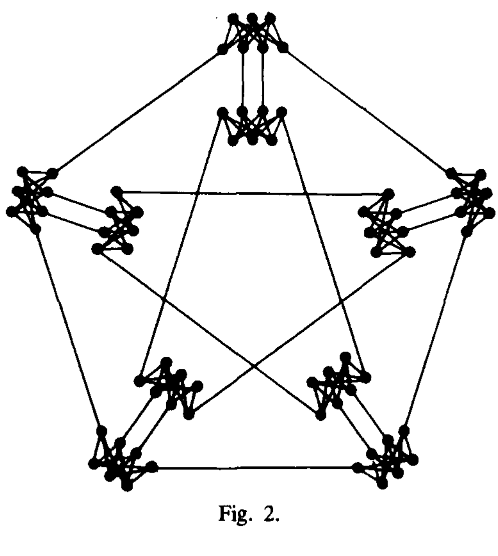

Fig. 2.

by choosing the Peterson graph for G, the "obvious" 1-factor for F and substituting for each vertex of G^* a copy of $K_{3,4}$.

LEMMA 4. *Let G be a 4-connected, 4-regular graph. Let $e_1 = (x_1, x_2)$ and $e_2 = (x_3, x_4)$ be two disjoint edges of G. Let G' be any 4-regular, 4-connected graph, $G \cap G' = \emptyset$. Let $g \in V(G)$ be any vertex and let $N(g) = \{g_1, g_2, g_3, g_4\}$. Let*

$$H = (G \setminus \{e_1, e_2\}) \cup (G' \setminus \{g\}) \cup \{(x_1, g_1), (x_2, g_2), (x_3, g_3), (x_4, g_4)\}.$$

H is a 4-regular 4-connected graph.

PROOF. Since G' is 4-connected, $G' \setminus \{g\}$ is 3-connected. By Menger's theorem, $G' \setminus \{g\}$ has a pair of disjoint $\{g_1, g_3\} - \{g_2, g_4\}$ paths. Without loss of generality, we may assume that one path, P_1, has endpoints $\{g_1, g_2\}$ and the other, P_2, has endpoints $\{g_3, g_4\}$. Obviously, H is 4-regular. To show that H is 4-connected, we will show that any pair of vertices is connected by four disjoint paths. Let $\{a, b\} \subseteq V(G)$. By replacing the edges e_1 and e_2 on any $a - b$ path, by the paths P_1 and P_2 respectively, any pair of disjoint $a - b$ paths will be transformed into disjoint paths. Since G is 4-connected, it contains four disjoint $a - b$ paths and therefore so does H. If $\{a, b\} \subseteq V(G' \setminus \{g\})$, let Q_1, Q_2, Q_3, Q_4 be four disjoint $a - b$ paths in G'. If $g \notin Q_i$ then $Q_i \in H$. If $g \in Q_1$, then we must have $Q_1 = (a, \cdots, g_i, g, g_j, \cdots, b)$. Since $G \setminus \{e_1, e_2\}$ is connected, it contains an $x_i - x_j$ path $(x_i, \cdots, x_j)$. If we replace the path Q_1 by the path $Q_1' = (a, \cdots, g_i, x_i, \cdots, x_j, g_j, \cdots, b)$ the four paths Q_1', Q_2, Q_3 and Q_4 are obviously four disjoint $a - b$ paths in H. If $a \in G' \setminus \{g\}$ and $b \in G$, let

$$P_i = (a, \cdots, g_i, g), \qquad i = 1, 2, 3, 4$$

be four disjoint $a - g$ paths in G'. If $b \neq x_i$, by a variant of Menger's theorem, G contains four disjoint $b - \{x_1, x_2, x_3, x_4\}$ paths $Q_i = (b, \cdots, x_i)$. It is easily seen that none of these paths contains the edge e_1 or e_2. The paths

$$\bar{P}_i = (a, \cdots, g_i, x_i, \cdots, b), \qquad i = 1, 2, 3, 4$$

obviously constitute four disjoint $a - b$ paths in H. Finally, if $b = x_1$, let $Q_i = (x_1, \cdots, x_j)$, $j = 2, 3, 4$ be three disjoint $x_1 - \{x_2, x_3, x_4\}$ paths in $G \setminus \{e_1\}$. Such paths exist since $G \setminus \{e_1\}$ is at least three connected. Again, none of these paths contains e_2. The four paths

$$\bar{P}_j = (a, \cdots, g_j, x_j, \cdots, x_1), \quad j = 2, 3, 4; \qquad \bar{P}_1 = (a, \cdots, g_1, x_1)$$

are four disjoint $a - b$ paths in H. We say that H is obtained from G by replacing $\{e_1, e_2\}$ by G'.

THEOREM 2. *Let G be a 3-connected cubic graph. Let $F \subseteq E(G)$ be a 1-factor of G and let $F_1 \subseteq F$ be a set of edges such that no Hamiltonian cycle of G contains F_1. Let $\bar{G}$ be obtained from G as in Theorem 1. Let $\bar{\bar{G}}$ be obtained from $\bar{G}$ by replacing each pair of edges of $\bar{G}$, derived from doubling the edges of F_1, by an arbitrary 4-regular 4-connected graph.* (See Fig. 3.) *Then $\bar{\bar{G}}$ is a 4-regular, 4-connected non-Hamiltonian graph.*

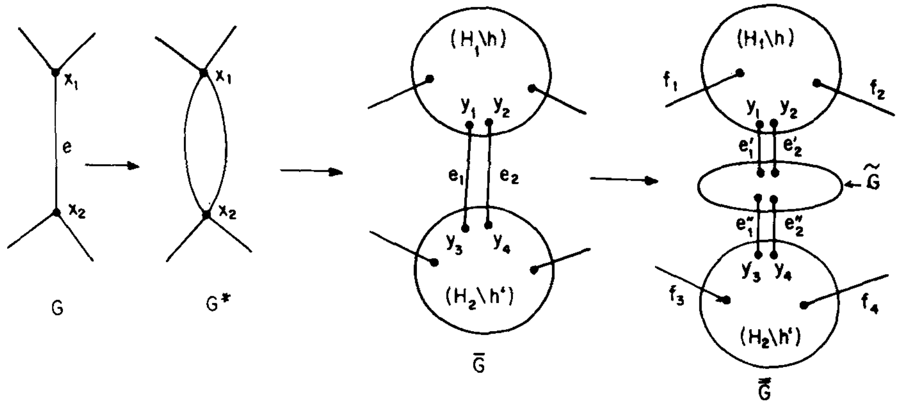

Fig. 3.

PROOF. If a cycle C in $\bar{\bar{G}}$ covers all vertices of $H_1 \backslash \{h\}$, and contains both edges $\{e'_1, e'_2\}$ (Fig. 3), then since $H_1 \in \mathcal{H}_4$, $C \cap \{f_1, f_2\} = \emptyset$. It follows that either $\{e''_1, e''_2\} \subseteq C$, or $\{e''_1, e''_2\} \cap C = \emptyset$. Therefore, C cannot be a Hamiltonian cycle in $\bar{\bar{G}}$. Again, since $H_i \in \mathcal{H}_4$, any Hamiltonian cycle of $\bar{\bar{G}}$ would have to intersect $H_i \backslash \{h\}$ and $\tilde{G}$ in a path. But then the contractions of these paths to a single vertex, and replacing the adjacent edges e'_i, e''_j by the edge e in G, yields a Hamiltonian cycle in G that contains F_1. This contradicts the choice of F_1 and the theorem is proved.

We denote by $\mathcal{M}$, the family of all 4-regular, 4-connected graphs obtained as in Theorem 2. Obviously, $\mathcal{M}_1 \subset \mathcal{M}$ $(F_1 = \emptyset)$. Figure 4 is a member of $\mathcal{M}$, with only 54 vertices. It is obtained by choosing G to be the triangular prism, and replacing the three pairs of edges (the "vertical" 1-factor) by copies of K_5.

In spite of the large number of graphs in $\mathcal{M}$, no member of $\mathcal{M}$ is 4-polytopal. To show that, we first obtain two types of forbidden subgraphs in a simple 4-polytopal graph. It is worthwhile mentioning that by Steinitz's theorem (and Kuratowski's characterization of planar graphs) a 3-connected cubic graph is 3-polytopal iff it does not contain a homomorphic image of $K_{3,3}$. Theorems 3 and 4 show that no such finite family of forbidden subgraphs exists for simple 4-polytopes.

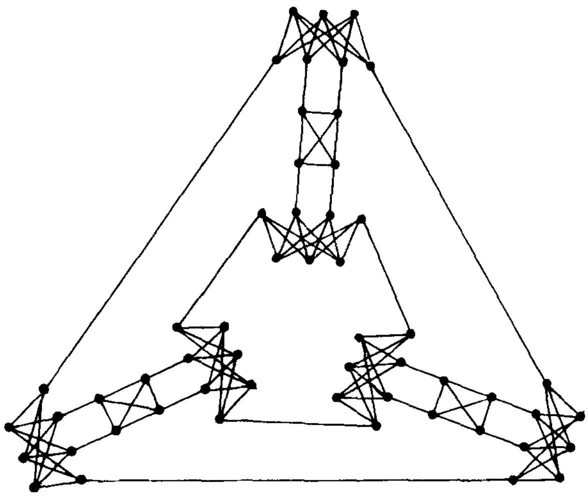

Fig. 4.

THEOREM 3. *If G is a 4-connected 4-regular graph, containing $K_{3,4}$ as a subgraph, then G is not polytopal.*

PROOF. Since G is 4-connected, it is easy to see that $K_{3,4}$ must be an induced subgraph. Obviously, G is not 3-polytopal. Consider the edge (b_4, c_4) in Fig. 5. By (1) and (2), (b_4, c_4) is contained in exactly 3 "facets" of G, F_1, F_2, F_3. Since F_i are cubic, b_4 must have at least two of the three vertices $\{a_1, a_2, a_3\}$ as neighbors in each F_i. Since $F_1 \cap F_2 \cap F_3 = (b_4, c_4)$ (the intersection is 1-dimensional) it is easy to see that no F_i can contain all three vertices $\{a_1, a_2, a_3\}$. Let F_4 be the fourth "facet" containing b_4. Since $c_4 \notin F_4$, $\{a_1, a_2, a_3\} \subseteq F_4$. Applying the above argument to any of the edges (a_i, b_i), $i = 1, 2, 3$, shows that F_4 can contain none of these edges. But then $F_4 \subseteq K_{3,4}$, and since F_4 is a cubic graph, and since the only cubic subgraph of $K_{3,4}$ is a $K_{3,3}$, F_4 cannot be planar.

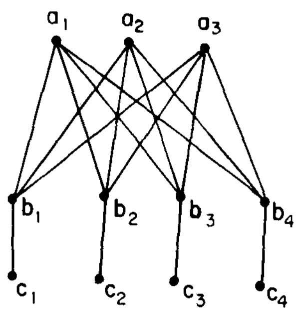

Fig. 5.

COROLLARY. *The Meredith graph is not 4-polytopal, neither is any 4-regular 4-connected graph in $\mathcal{M}$ in which $K_{3,4}$ is used for a substitution.*

THEOREM 4. *Let G be a 4-connected 4-regular graph containing a subgraph as described in Fig. 6. (We assume that $\{e_1, e_2, f_1, f_2\}$ and $\{e_1, e_2, f_3, f_4\}$ are the only edges incident with G_1 and G_2, respectively, and f_1, f_2, f_3, f_4 are 4-distinct edges.) Then G is not 4-polytopal.*

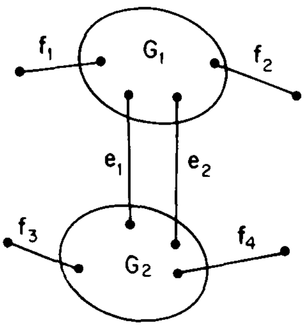

Fig. 6.

PROOF. If $\{e_1, e_2\}$ are not contained in the same "facet", then the three "facets" containing e_1 will have to contain the edges $\{e_1, f_1, f_2\}$ which is impossible. Hence we may assume that G has a "facet" $F \supseteq \{e_1, e_2\}$. F is a cubic, 3-connected subgraph of G, hence $F \cap \{f_1, f_2\} \neq \emptyset$. e_1 is the intersection of exactly 3 "facets". Hence we have a "facet" F' such that $e_1 \in F'$, but $e_2 \notin F'$, so we must have $F' \supseteq \{f_1, f_2\}$. Since $F' \cap F$ is a cycle containing e_1, we cannot have $\{f_1, f_2\} \subseteq F$. Without loss of generality, we may assume that $f_2 \notin F$. Applying the same argument to the face F and the subgraph G_2, we obtain that $f_3 \in F$ but $f_4 \notin F$. Since $f_3 \neq f_1$, the set $\{f_1, f_3\}$ is an edge cut set of F; this contradicts the 3-connectivity of F.

COROLLARY. *No graph in $\mathcal{M}$ is 4-polytopal.*

REMARKS. In spite of the fact that the extensive family of 4-connected 4-regular non-Hamiltonian graphs $\mathcal{M}$ fails to even "look like" 4-polytopal graphs, the author still believes (though with less certainty) that Barnette's conjecture is false. We definitely need new constructions of non-Hamiltonian, 4-regular, 4-connected graphs to justify that.

From a "graphical" point of view, the family $\mathcal{M}$ can be generalized to yield $2r$-regular, $2r$-connected non-Hamiltonian graphs. These graphs yield new upper bounds on the cyclability of k-regular, k-connected graphs. For example, the graph in Fig. 4 has 21 vertices that are not contained in a cycle. For more details on this subject, see W. D. Mccuaig and M. Rosenfeld [5].

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